The Fokker-Planck equation
[fɔkə] [plɑenk]

28 March 07

$P(v,t \mid v_0 t_0)dv$: probabity finding the Brownian particles have velocity $[v, v+dv]$ given that the particle is at v_0 at time t_0

Consider stationary process, then

$$P(v,t \mid v_0 t_0) = P(v, t-t_0 \mid, v_0, 0)$$

i.e. we consider that v_0 is the value at time $t=0$ and write $P(v,t \mid v_0)$

$$\int_{-\infty}^{\infty} P(v,t \mid v_0)\,dv = 1$$

$$\lim_{t\to 0} P(v,t \mid v_0) = \delta(v - v_0)$$

(initial condition)

for $t \to \infty$ system forget its history v_0
velocity should follow Maxwell distribution

$$\lim_{t\to +\infty} P(v,t, \mid v_0) \propto e^{-\beta(\frac{1}{2}mv^2)}$$

with proper normalization $= \left(\frac{m\beta}{2\pi}\right)^{\frac{1}{2}} e^{-\beta(\frac{1}{2}mv^2)}$

Associated Langevin Eq is

$$m\frac{dv}{dt} = -m\gamma v + R(t)$$

F. Reif Chap 15
R. Zwanzig Chap 1 & 2

Kramers-Moyal expansion

master Eq. τ small infinitisimally moving a way from current v

$$\frac{\partial P(v,t|v_0)}{\partial t} dv \cdot \tau = -\int_{v'} P(v,t|v_0)\, P(v',\tau|v)\, dv\, dv' +$$

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rate of increase in prob finding particle in $[v, v+dv]$

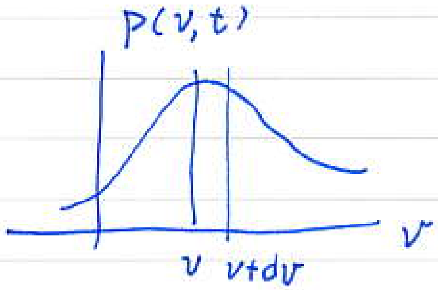
after time elapsed by τ the prob. that particle has velocity v' is $P(v',\tau|v)dv'$

$$\approx [P(v,t+\tau|v_0) - P(v,t|v_0)]dv$$

$$P(A)\,P(B|A) = P(A \cap B)$$

$$+ \int_{v'} P(v',t|v_0)\,dv'\; P(v,\tau|v')\,dv$$

another velocity v' can be come the current one.



but $\int_{v'} P(v',\tau|v)\,dv' = 1$

$v' = v - \xi$ ← small change

(see derivation by Reif)

$$\frac{\partial P(v,t|v_0)}{\partial t}\tau = -P(v,t|v_0) + \int_{-\infty}^{\infty} P(v-\xi,t|v_0)\,P(v,\tau|v-\xi)\,d\xi$$

$\tau \to 0$. ξ small. P change slowly with respect to v $\quad X_0 = v$

$X = v - \xi = X_0 - \xi$

X variable fix ξ

$$P(v-\xi,t|v_0)\,P(v,\tau|v-\xi)$$

$$= P(v-\xi,t|v_0)\,P(v-\xi+\xi,\tau|v-\xi) = P(X,t|v_0)\,P(X+\xi,\tau|X)$$

$$= P(X_0,t|v_0)\,P(X_0+\xi,\tau|X_0) + \frac{\partial}{\partial X} P(X,t|v_0)\,P(X+\xi,\tau|X)\Big|_{X=v}(X-X_0)$$

$$= P(v,t|v_0)\,P(v+\xi,\tau|v) + (-\xi)\frac{\partial}{\partial v}\left(P(v,t|v_0)\,P(v+\xi,\tau|v)\right)$$

$$+\frac{1}{2}(-\xi)^2\frac{\partial^2}{\partial v^2}P(v,t|v_0)\,P(v+\xi,\tau|v) + \cdots$$

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$$\frac{\partial P(v,t|v_0)}{\partial t}\tau = -P(v,t|v_0) +$$

cancel

$$\int_{-\infty}^{\infty} P(v,t|v_0)\,P(v+\xi,\tau|v)\,d\xi$$

$$+\frac{\partial}{\partial v}P(v,t|v_0)\int_{-\infty}^{\infty}(-\xi)\,P(v+\xi,\tau|v)\,d\xi$$

$$\frac{1}{2}\frac{\partial^2}{\partial v^2}P(v,t|v_0)\int_{-\infty}^{\infty}\xi^2\,P(v+\xi,\tau|v)\,d\xi + \cdots$$

$$v+\xi = v' = v(\tau)$$

$$M_n = \frac{1}{\tau}\int_{-\infty}^{\infty}\xi^n\,P(v+\xi,\tau|v)\,d\xi = \frac{1}{\tau}\langle[v(\tau)-v]^n\rangle$$

change of velocity after time τ

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial v}(M_1 P) + \frac{1}{2}\frac{\partial^2}{\partial v^2}(M_2 P)$$ ← Fokker-Planck Eq.

$$M_1 = \frac{1}{\tau}\langle(v(\tau)-v)\rangle = \frac{1}{\tau}\left[\langle v(\tau)\rangle - v\right] \qquad v(\tau=0)=v$$

avge velocity

$$\langle v(\tau)\rangle = v\,e^{-\gamma\tau} = v\left[1-\gamma\tau+O(\tau^2)\right]$$

$$= v - \gamma v\tau + \cdots$$

$$= -\gamma v$$

$$M_2 = \frac{1}{\tau}\langle(v(\tau)-v)^2\rangle = \frac{2k_BT}{m}\gamma \qquad = 2D\gamma^2$$

$M_n = 0$ for $n \geq 3$ for gaussian process!

$$v(t) = v(0)\, e^{-\gamma t} + \int_0^t \frac{R(\tau)}{m} e^{-\gamma(t-\tau)} d\tau$$

τ small

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$$\lim_{t\to 0} \frac{1}{t}\langle (v(t) - v(0))^2 \rangle = \left\langle \left[v(0)(e^{-\gamma t} - 1) + \int_0^t \frac{R(\tau)}{m} e^{-\gamma(t-\tau)} d\tau \right]^2 \right\rangle$$

$$= \frac{1}{t}\left[\underbrace{\langle (v(0)[-\gamma t + \cdots])^2 \rangle}_{0 \text{ at } t\to 0} + \langle R(\tau) R(\tau') \rangle \frac{1}{m^2} \right]$$

No cross term

$$= \frac{1}{t} \frac{C \cdot t}{m^2} \qquad C = 2m\gamma k_B T \qquad \int_0^t \int_0^t$$

→ with spectrum M_1, M_2
we have

$$\frac{\partial P}{\partial t} = \gamma \frac{\partial}{\partial v}(vP) + \underbrace{\gamma \frac{k_B T}{m}}_{\gamma^2 D} \frac{\partial^2 P}{\partial v^2}$$

$$D = \frac{k_B T}{m\gamma}$$

solution of Fokker-Planck equation

γ v

follow F. Reif

Let $\frac{\partial^2 P}{\partial v^2} = 0$ first

$$\frac{\partial P}{\partial t} - \gamma v \frac{\partial P}{\partial v} = \gamma P$$

multiply by $\lambda(v, t)$ so that

$$\lambda \frac{\partial P}{\partial t} - \lambda \gamma v \frac{\partial P}{\partial v} = \lambda \gamma P$$ is a total differential

$$\frac{\partial P}{\partial t} dt + \frac{\partial P}{\partial v} dv = dP$$

we identify

$$\left.\begin{aligned} dt &= \lambda \\ dv &= -\lambda \gamma v \\ dP &= \lambda \gamma P \end{aligned}\right\} \qquad \begin{aligned} \frac{dv}{dt} &= -\gamma v \\ \frac{dP}{dt} &= \gamma P \end{aligned}$$

$$v = u e^{-\gamma t}, \quad P = Q e^{\gamma t}$$

Now add in the diffus term

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Let $P(v,t) = Q(u,t)\, e^{\gamma t}$ $\qquad u = v e^{\gamma t}$

→ new velocity-like variables

$$\frac{\partial P}{\partial v} = e^{\gamma t}\frac{\partial Q}{\partial u}\frac{\partial u}{\partial v} = e^{2\gamma t}\frac{\partial Q}{\partial u}$$

$$\frac{\partial^2 P}{\partial v^2} = e^{2\gamma t}\frac{\partial^2 Q}{\partial u^2}\frac{\partial u}{\partial v} = e^{3\gamma t}\frac{\partial^2 Q}{\partial u^2}$$

$$\frac{\partial P}{\partial t} - \gamma v \frac{\partial P}{\partial v} - \gamma P = \frac{\partial^2 P}{\partial v^2}\left(\gamma \frac{k_B T}{m}\right)$$

↓

$$\left.\frac{\partial Q}{\partial t}\right|_u e^{\gamma t} + \cancel{Q\gamma e^{\gamma t}} - \gamma v e^{2\gamma t}\frac{\partial Q}{\partial u} - \cancel{\gamma Q e^{\gamma t}} = e^{3\gamma t}\frac{\partial^2 Q}{\partial u^2}\left(\gamma\frac{kT}{m}\right)$$

$$+ e^{\gamma t}\left.\frac{\partial Q}{\partial u}\right|_t \frac{\partial u}{\partial t} \leftarrow v\gamma e^{\gamma t}$$

$$\tilde{D} = \frac{\gamma k_B T}{m}$$

$$\frac{\partial Q}{\partial t} = e^{2\gamma t}\frac{\partial^2 Q}{\partial u^2}\left(\gamma\frac{k_B T}{m}\right) \rightarrow \frac{\partial Q}{e^{2\gamma t}\partial t} = \left(\gamma\frac{k_B T}{m}\right)\frac{\partial^2 Q}{\partial u^2}$$

Let $d\theta = e^{2\gamma t}\,dt \Rightarrow \theta = \frac{1}{2\gamma}\left(e^{2\gamma t}-1\right)$

$\theta \approx t$ when t small

$$\frac{\partial Q}{\partial \theta} = \tilde{D}\frac{\partial^2 Q}{\partial u^2} \rightarrow \text{standard diffus eq.}$$

$$Q = \frac{1}{\sqrt{4\pi\tilde{D}\theta}}\, e^{-\frac{(u-u_0)^2}{4\tilde{D}\theta}} \qquad Q \to \delta(u-u_0) \text{ as } \theta \to 0$$

$$P(v,t|v_0) = e^{\gamma t}\frac{1}{\sqrt{4\pi\tilde{D}\frac{1}{2\gamma}\left(e^{2\gamma t}-1\right)}}\exp\left[-\frac{\left(v e^{\gamma t}-v_0\right)^2}{4\tilde{D}\frac{1}{2\gamma}\left(e^{2\gamma t}-1\right)}\right]$$

$$= \left[\frac{m}{2\pi k_B T (1 - e^{-2\gamma t})} \right]^{\frac{1}{2}} \exp\left[- \frac{m(v - v_0 e^{-\gamma t})^2}{2 k_B T (1 - e^{-2\gamma t})} \right]$$

$t \to \infty$ reduce to Maxwell distribution

$$\frac{\partial P}{\partial t} = \gamma \frac{\partial}{\partial v}(v P) + \frac{\gamma k_B T}{m} \frac{\partial^2 P}{\partial v^2}$$

equilibrium distribution 1st $t \to \infty$ P doesn't change with time

$$\frac{\partial}{\partial v}(vP) + \frac{k_B T}{m} \frac{\partial^2}{\partial v} P = 0$$

or $\quad vP + \frac{k_B T}{m} \frac{\partial P}{\partial v} = \text{const.}$

const must equal 0. before $\int_{-\infty}^{+\infty} P \, dv = 1$

integrate on both side $\quad \left| \int_{-\infty}^{+\infty} v P \, dv \right| < M$ bounded

$\int_{-\infty}^{+\infty} \text{const} \, dv$ must be finite

$$-\frac{m v \, dv}{k_B T} \equiv \frac{\partial P}{P}$$

$$\to P = P_0 e^{-\beta\left(\frac{1}{2} m v^2\right)} \qquad \beta = \frac{1}{k_B T}$$

$\leftarrow$ Maxwell velocity distribution

From Langevin to Fokker-Planck, general setting

$$\underset{\text{vector}\,\to}{\vec{\dot{X}}} = G(X) + \underset{\text{matrix}}{g(X)}\, \xi \qquad \langle \xi \xi^T \rangle = 2D\, \delta(t - t')$$

D is const

$$\frac{\partial P}{\partial t} = - \underset{\text{row vector}}{\frac{\partial}{\partial X}^{T}} G P + \left(\frac{\partial}{\partial X}\right)^T g\, D \left[\overleftrightarrow{g^T} \frac{\partial}{\partial X} P \right]$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$ is arbitrary vector

Kinetic theory

entropy of classical gas

$$S = Nk_B\left\{\ln\left(\frac{V}{N\lambda^3}\right) + \frac{5}{2}\right\}$$

when gas can be treated classically

$$\lambda = \frac{h}{\sqrt{2\pi m k_B T}} \qquad \frac{\lambda}{\left(\frac{V}{N}\right)^{1/3}} = \frac{\lambda}{b} \ll 1$$

b ← average space of particle

O_2

$$\lambda = \frac{10^{-34}}{\sqrt{10^{-26}\, 1.38\times10^{-23} \times 300\,K}} = 1.5\times10^{-11}\,m = 0.15\,\text{Å}$$

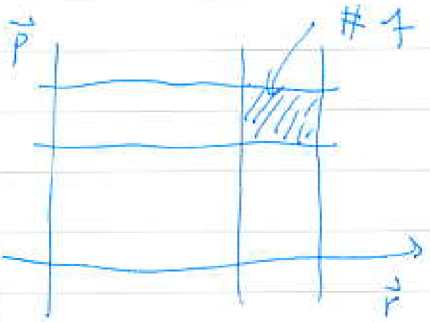
dilute gas case

$b \approx 10 - 30$ Å

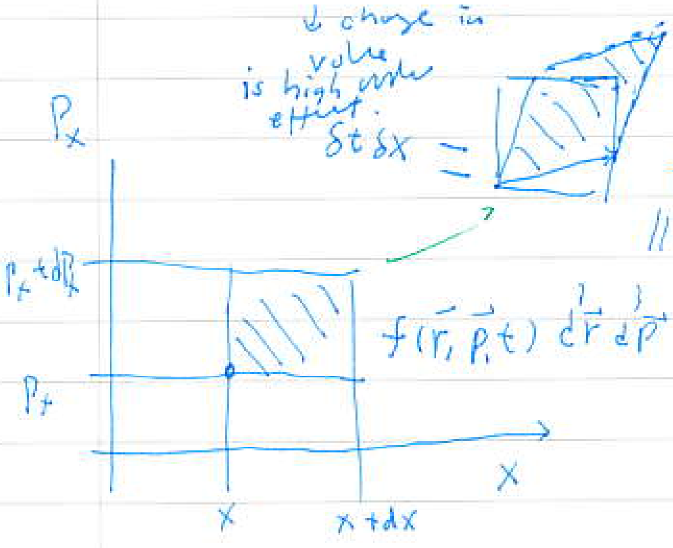
$\lambda \uparrow$ is not ok

$f(\vec r, \vec p, t)\, d^3\vec r\, d^3\vec p$ = # of molecules having position $\vec r$ in volume $d^3\vec r$ & momentum $\vec p$ .. $d^3\vec p$

joint distribution makes sense only in classical regime



$$\int f(\vec r, \vec p, t)\, d^3\vec r\, d^3\vec p = N$$



volume is preserved provided a single particle system is Hamiltonian system.

$$f(\vec r + \vec v\,\delta t,\ \vec p + F\delta t,\ t + \delta t)\, d^3\vec r\,' d^3\vec p\,'$$

$$\parallel$$

$$f(\vec r, \vec p, t)\, d^3\vec r\, d^3\vec p$$

volume do not change (some of Liouville theorem)

external force only

$$\frac{Df}{Dt} \quad \frac{\partial f}{\partial t} = \left(\frac{\partial}{\partial t} + \frac{\vec p}{m}\cdot\nabla_r + \vec F\cdot\nabla_p\right) f = \left(\frac{\partial f}{\partial t}\right)_{coll}$$

$$f(\vec r, \vec p) = N\int \rho(\vec r, \vec p; \vec r_2 \vec p_2\, \vec r_3 \vec r_4 \ldots)\, d\vec r_2 d\vec p_2 \ldots d\vec r_N d\vec p_N$$

$$\sum_i \int \rho(\ \ldots\ )$$

sum of 1st particle in $d\vec r\, d\vec p$, 2nd in $d\vec r\, d\vec p$, ... etc.

streaming

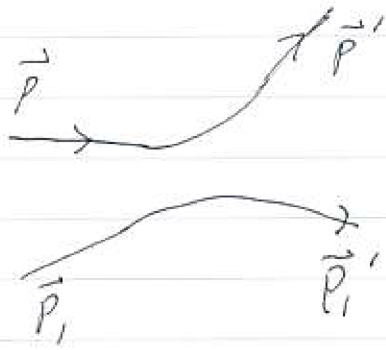
$$\frac{Df}{Dt} = \left(\frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla}_r + \vec{F}\cdot\vec{\nabla}\right)f = \left(\frac{\partial f}{\partial t}\right)_{coli}$$

$$= \bar{R} - R$$

R # of collisions occuring ~~during~~ per unit time in which one of the initial molecules

$R\,\delta t\, d^3\vec{r}\, d^3\vec{p}$ = no of collisons occuring during the time between t and $t+\delta t$ in which one of the initial molecules is in $d^3\vec{r}\, d^3\vec{p}$ abt $(\vec{r}, \vec{p})$

$\bar{R}\,\delta t\, d^3\vec{r}\, d^3\vec{p}$ = no. of collisions occuring during the time between t and $t+\delta t$ in which one of the final molecules is in $d^3\vec{r}\, d^3\vec{p}$ abt $(\vec{r}, \vec{p})$.

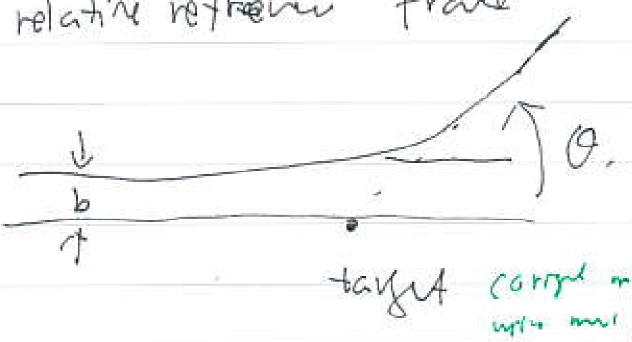


$$\vec{p} + \vec{p}_1 = \vec{p}\,' + \vec{p}_1\,'$$

$$\varepsilon(\vec{p}) + \varepsilon(\vec{p}_1) = \varepsilon(\vec{p}\,') + \varepsilon(\vec{p}_1\,')$$

$$\varepsilon = \frac{p^2}{2m}$$

in relative reference frame



given $\vec{p}$, $\vec{p}_1$

b, ϕ or (θ, ϕ) can still vary.

$$(p, p_1|\sigma|p', p_1') \propto \delta(p + p_1 - p' - p_1')\,\delta[\varepsilon(p) + \varepsilon(p_1) - \varepsilon(p') - \varepsilon(p_1')]$$

differential cross-section

target fixed, incoming flux is I

$I\,\sigma(\theta, \varphi)\,d\Omega$ = # of of incident molecules scattered per second into solid angle $d\Omega$ in direct (θ, φ).

$$\left(\frac{\partial f}{\partial t}\right)_{coll} = -\int d\vec{P}_1 \int d\vec{P}_1{}' \int d\vec{P}{}' \, \sigma(P, P_1 | P' P_1') (f f_1 - f' f_1')$$

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1st is R 2nd is $\bar{R}$

$$f_1 = f(r, P_1 t)$$
$$f' = f(r, P', t)$$
$$f_1' = f(r, P_1', t)$$

another forme

$$= -\int d\vec{P}_1 \, d\Omega \, |v_1 - v| \, \sigma(\theta, \varphi) (f f_1 - f' f_1')$$

$= 4 \times 3$

~~$\vec{P}_1 \vec{P}_1$~~ $\vec{P}, \vec{P}_1, \vec{P}', \vec{P}_1'$ have 12 degrees of freedom

conservation law for P & E take away 4 degree of freedom

still left with 8 degrees of freedom which is

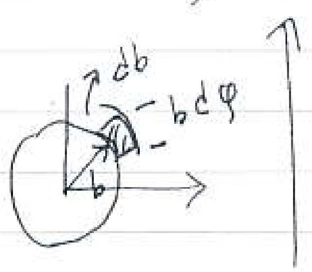
$\vec{P}, \vec{P}_1, d\Omega \leftarrow$ out going angles in relative reference frame.

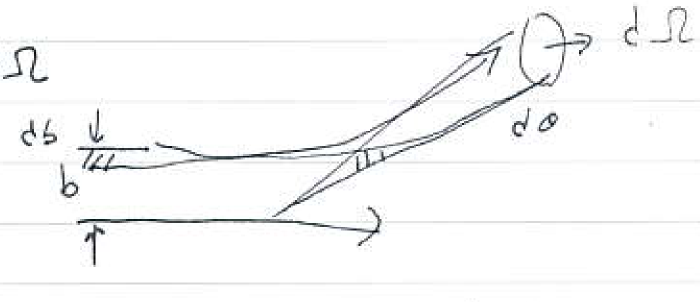
$3 + 3 + 2$

Boltzmann b: impact parameter: we can think of this equation as precisely define σ.

Important relation betw σ and impact parameter b

$$d\omega = b \, db \, d\varphi = \sigma \, d\Omega$$





this equation simply says # of particles come from the area $db \cdot b d\varphi$ must go out to solid angle $d\Omega$, which is $\sigma d\Omega$ by definition

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Two-body scattering problem

$$\mathcal{H}(\vec{r}_1, \vec{r}_2, \vec{p}_1, \vec{p}_2) = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(|\vec{r}_1 - \vec{r}_2|)$$

reduce to one-body problem

$$\begin{cases} \vec{r} = \vec{r}_2 - \vec{r}_1 & \text{relative position} \\ \vec{R} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} & \text{Center of mass} \end{cases}$$

moment of the Center of mass

$$\vec{P} = M\dot{\vec{R}} = (m_1+m_2)\dot{\vec{R}}$$

$$\mu = \frac{m_1 m_2}{m_1+m_2}, \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

solve $\vec{r}_1$ & $\vec{r}_2$ in terms $\vec{r}$ & $\vec{R}$

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2(\vec{r}+\vec{r}_1)}{m_1+m_2} \qquad (m_1+m_2)\vec{R} = (m_1+m_2)\vec{r}_1 + m_2\vec{r}$$

$$\vec{r}_1 = \vec{R} - \frac{m_2}{m_1+m_2}\vec{r}, \qquad \vec{r}_2 = \vec{r} + \vec{r}_1 = \vec{R} + \frac{m_1}{m_1+m_2}\vec{r}$$

$$\vec{p}_1 = m_1\dot{\vec{r}}_1 = m_1\dot{\vec{R}} - \frac{m_1 m_2}{m_1+m_2}\dot{\vec{r}} = m_1\dot{\vec{R}} - \mu\dot{\vec{r}}$$

$$\vec{p}_2 = m_2\dot{\vec{r}}_2 = m_2\dot{\vec{R}} + \frac{m_1 m_2}{m_1+m_2}\dot{\vec{r}} = m_2\dot{\vec{R}} + \mu\dot{\vec{r}}$$

$$\vec{p}_1 + \vec{p}_2 = \vec{P} = \text{const.} \qquad \dot{\vec{r}} = v$$

$$\frac{\vec{p}_1^{\,2}}{2m_1} = \frac{1}{2m_1}\left(m_1^2\dot{\vec{R}}^2 + \mu^2\dot{\vec{r}}^2 - 2m_1\mu\dot{\vec{R}}\cdot\dot{\vec{r}}\right)$$

$$\frac{\vec{p}_2^{\,2}}{2m_2} = \frac{1}{2m_2}\left(m_2^2\dot{\vec{R}}^2 + \mu^2\dot{\vec{r}}^2 + 2m_2\mu\dot{\vec{R}}\cdot\dot{\vec{r}}\right)$$

$$= \frac{1}{2}\underbrace{(m_1+m_2)}_{M}\dot{\vec{R}}^2 + \frac{\mu^2}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)v^2$$

$$= \frac{1}{2M}\dot{\vec{R}}^2 + \frac{1}{2}[\mu]\dot{\vec{r}}^2$$

shown.

The collisn term in Boltzmann Eq.

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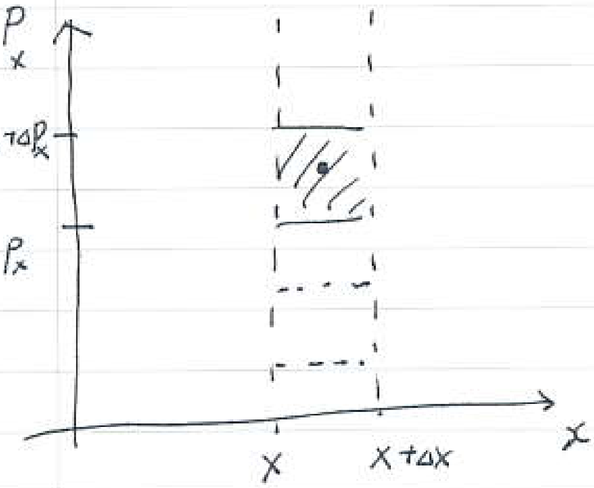


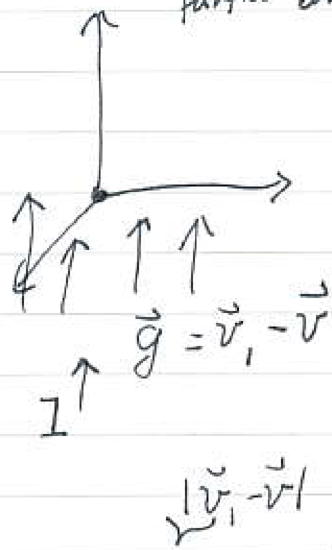
sizes of molecules are much smaller than Δx
each $d^3\vec{r}$ still has a lot of molecules ($\sigma_{tot} \ll \Delta x^2$)

molecules inside $d^3\vec{r}\, d^3\vec{p}$ do not collide as they have the same velocity

$f \cdot d\vec{p} = n$ is particle density

consider One molecule in $d^3\vec{r}\, d^3\vec{p}$.
and all the molecules in $d^3\vec{r}\, d^3\vec{p}_1$.
there is the flux of

$$I = f(\vec{r}, \vec{p}_1, t)\, |\vec{v}_1 - \vec{v}| \cdot d^3\vec{p}_1$$



the # of particles collided with the one (per second) is

$$I\, d\sigma = I\, \sigma(\theta, \varphi)\, d\Omega = f(\vec{r}, \vec{p}_1, t)\, \sigma(\theta, \varphi)\, d\Omega\, d\vec{p}_1$$

since there are $f(\vec{r}, \vec{p})\, d\vec{p}$ molecules there

The total collisn per second is (in unit $d^3\vec{r}$ and $d^3\vec{p}$)

$$I\sigma \cdot f(\vec{r}, \vec{p}) = f f_1\, \sigma\, |\vec{v}_1 - \vec{v}|\, d\Omega\, d\vec{p}_1$$

sum over all possible $\vec{p}_1$ and out come $d\Omega$.

$$\left(\frac{\partial f}{\partial t}\right)_{coli} = \text{\# of collisn per unit time in unit space and momen volumes.}$$

Assignment 5

$$G = G_0 + a(T-T^*)M^2 - \frac{1}{3}cM^3 + \frac{1}{4}dM^4$$

spontaneous M ← cubic eq.

(a) $0 = h = \frac{\partial G}{\partial M} = 2a(T-T^*)M - cM^2 + dM^3 = 0$ or $M = 0$

$$2a(T-T^*) - cM + dM^2 = 0$$

$$M = \frac{c \pm \sqrt{c^2 - 4d(T-T^*)\cdot 2a}}{2d}$$

$$= \frac{c \pm \sqrt{c^2 - 8ad(T-T^*)}}{2d}$$

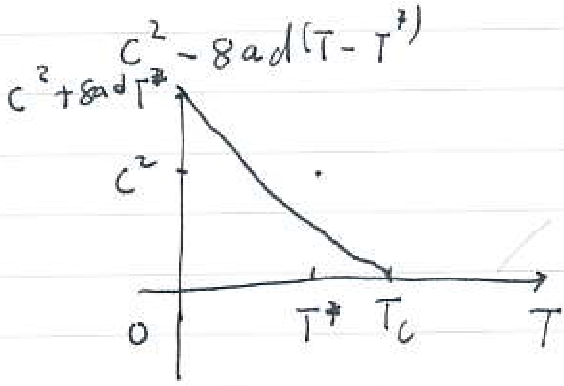
No real solutions if $c^2 - 8ad(T-T^*) < 0$

or $c^2 < 8ad(T-T^*)$

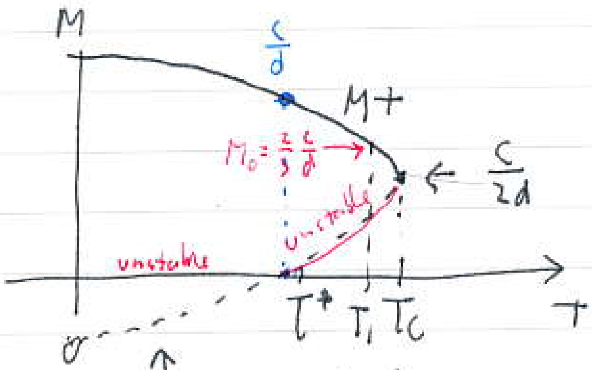
or $\frac{c^2}{8ad} < T - T^*$

or $T_c = \frac{c^2}{8ad} + T^* < T$.

So $T - T^* < \frac{c^2}{8ad}$



There are two solutions for M



other solutions unstable

$$2a(T-T^*) = dM^2 - cM$$

$$= d\left(M - \frac{c}{2d}\right)^2 - \frac{c^2}{4d^2}$$

need to be more ~~clear~~ careful

$M > 0$ has smaller G thus more stable

$$T_1 = T^* + \frac{1}{9}\frac{c^2}{d \cdot a}$$

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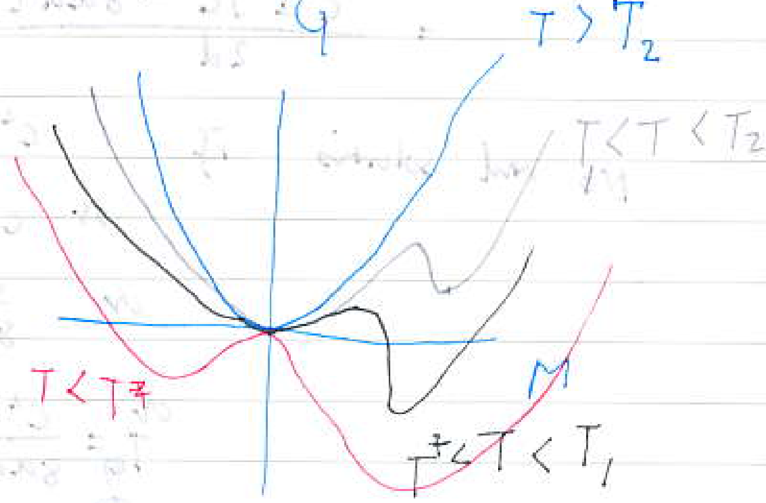
$$Q = T_1 \Delta S = T_1 \left(+0 - aM_0^2\right)$$

$$= \left(T^* + \frac{1}{9}\frac{c^2}{d \cdot a}\right)\left(a \cdot \left(\frac{2c}{3d}\right)^2\right)$$

$$= T_1 \, a \, \frac{c^2}{d^2}\left(\frac{4}{9}\right)$$



latent heat $Q = T_c \Delta S$ $= T_1 \Delta S$ $dQ = TdS$

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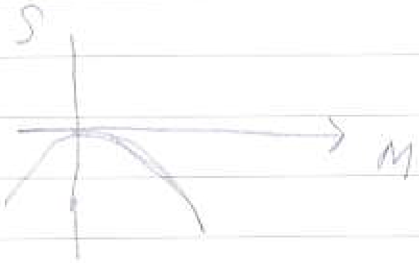
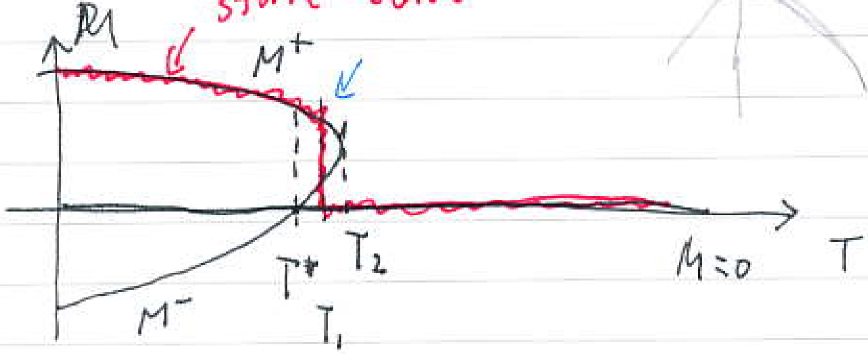
$$S = -\frac{\partial G}{\partial T} = -aM^2$$

$$\Delta S = S(T_c^+) - S(T_c^-) = 0 + aM^2 = a\left(\frac{c}{2d}\right)^2$$

$$Q = \left(\frac{c^2}{8ad} + T^*\right) \cdot a\left(\frac{c}{2d}\right)^2$$



solution



we have 3 solutions need to decide which one has global min.

$M=0$ solution $G = G_0$ indept of T

for $M \neq 0$ $G(M) = G(M=0)$

$$G_0 + a(T-T^*)M_0^2 - \frac{1}{3}cM_0^3 + \frac{1}{4}dM_0^4 = G_0$$

value of M at the transition T_1

$$\begin{cases} a(T-T^*) - \frac{1}{3}cM_0 + \frac{1}{4}dM_0^2 = 0 \\ 2a(T-T^*) - cM_0 + dM_0^2 = 0 \end{cases}$$

been M_0 is also on the M curve

but

eliminate $T-T^*$ $\left(-\frac{2}{3}+1\right)cM_0 + \left(\frac{1}{2} - 1\right)dM_0^2 = 0$

$$\frac{1}{3}c\cancel{M_0} - \frac{1}{2}dM_0^{\cancel{2}} = 0$$

$$\boxed{M_0 = \frac{2}{3}\frac{c}{d}}$$

$\frac{1}{9}\frac{c^2}{d}$

$$a(T-T^*) = \frac{1}{3}c\,\frac{2}{3}\frac{c}{d} - \frac{1}{4}d\left(\frac{2}{3}\frac{c}{d}\right)^2 = \left(\frac{2}{9} - \frac{4}{4\cdot 9}\right) =$$

low T expans

$$M = \langle \sum_{i=1}^{N} \sigma_i \rangle =$$

ground state	1st excited	
+ + + + + + + + + + + + + + +	+ + + + + + + + + − + + + + + + + + + +	+ + + + + + + + + + + − − + + + + + + +
$\sum \sigma_i = N$	$\sum \sigma_i = N-2$	$\sum \sigma_i = N-4$
$\Delta E = E - E_0 = 0$	$\Delta E = 8J$	$\Delta E = 12J$
degenerate 1	N	$2N$

Date

$$M = \frac{N \cdot 1 \cdot 1 + (N-2) e^{-8K} \cdot N + (N-4) e^{-12K} 2N}{1 + e^{-8K} N + 1 \cancel{N-4}\, e^{-12K} 2N} \qquad x = e^{-2K}$$

$$= \frac{N + N(N-2) x^4 + (N-4) \cdot 2N x^6}{1 + N x^4 + 2N x^6 + \cdots}$$

$$\frac{1}{1+x} = 1 - x + x^2 + \cdots$$

$$\cancel{= N + N}$$

$$\frac{M}{N} = \left[1 + (N-2) x^4 + 2(N-4) x^6\right]\left[1 - N x^4 - 2N x^6 + (N x^4 + 2N x^6)^2 + \cdots\right]$$

$$= 1 + (N-2) x^4 + 2(N-4) x^6 - N x^4 - 2N x^6 + O(x^8)$$

$$= 1 - 2x^4 - 8x^6 + \cdots$$

$$\sinh(2K) = \frac{e^{2K} - e^{-2K}}{2} = \left(\frac{1}{x} - x\right)\frac{1}{2}$$

$$\frac{M}{N} = \left(1 - \frac{1}{\sinh^4(2K)}\right)^{\frac{1}{8}}$$

$$= \left(1 - \frac{1}{\frac{1}{16}\left(\frac{1}{x} - x\right)^4}\right)^{\frac{1}{8}}$$

$$= \left(1 - \frac{16}{\left[\frac{1}{x}(1 - x^2)\right]^4}\right)^{\frac{1}{8}} = \left(1 - \frac{16 x^4}{(1 - x^2)^4}\right)^{\frac{1}{8}}$$

$$= \left(1 - 16 x^4 (1 + x^2 + x^4 + x^6 + \cdots)^4\right)^{\frac{1}{8}}$$

$$\frac{M}{N} = \left(1 - 16x^4(1 + 4x^2 + \cdots)\right)^{\frac{1}{8}} \qquad x = e^{-2K}$$

$$= \left(1 - 16x^4 - 16\cdot 4x^6 + \cdots\right)^{\frac{1}{8}}$$

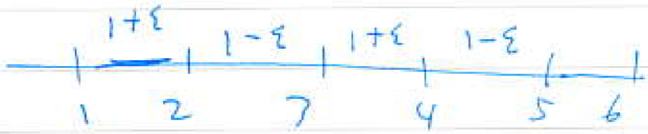
Date

$$= 1 - \frac{1}{8}\cdot 16x^4 - \frac{16}{8}\cdot 4x^6 + \cdots = 1 - 2x^4 - 8x^6 + \cdots$$

$$-34x^8$$

$$-152x^{10}$$

2. (a) $$Z_\sigma = \sum_{\{\sigma\}} e^{-\beta \sum_{i=1}^{N}(1+\varepsilon(-1)^i)\sigma_i\sigma_{i+1}}$$



$$\underbrace{e^{\beta\sigma_1\sigma_2(1+\varepsilon)}}_{P}\ \underbrace{e^{\beta\sigma_2\sigma_3(1-\varepsilon)}}_{Q}\ e^{\sigma_3\sigma_4(1+\varepsilon)}\ e^{\cdots}$$

$$P = \begin{pmatrix} e^{\beta(1+\varepsilon)} & e^{-\beta(1+\varepsilon)} \\ e^{-\beta(1+\varepsilon)} & e^{\beta(1+\varepsilon)} \end{pmatrix} \qquad Q = \begin{pmatrix} e^{+\beta(1-\varepsilon)} & e^{-\beta(1-\varepsilon)} \\ e^{-\beta(1-\varepsilon)} & e^{+\beta(1-\varepsilon)} \end{pmatrix}$$

(b)

$$\cancel{F_\sigma = -k_B T}$$

$$Z_\sigma = \mathrm{Tr}(PQ)^{\frac{N}{2}} = \cancel{\frac{N}{2}\ln\lambda}\ \lambda^{\frac{N}{2}}$$

$$Z_{tot} = \int_{-\infty}^{+\infty} d\varepsilon\, e^{-\beta N w\varepsilon^2 + \frac{N}{2}\ln\lambda}$$

$$= \int_{-\infty}^{\infty} d\varepsilon\, e^{-\beta N\underbrace{\left(w\varepsilon^2 + \frac{-1}{2\beta}\ln\lambda\right)}_{g(\varepsilon)}}$$

$$g(\varepsilon) = w\varepsilon^2 - \frac{1}{2}k_B T\ln\lambda$$

let $x = e^{\beta}$

$y = e^{\beta\varepsilon}$

$$PQ = \begin{pmatrix} x\cdot y & \frac{1}{xy} \\ \frac{1}{xy} & x\cdot y \end{pmatrix}\begin{pmatrix} x/y & y/x \\ y/x & x/y \end{pmatrix}$$

$a = x^2 + \frac{1}{x^2}$

$b = y^2 + \frac{1}{y^2}$

$$= \begin{pmatrix} x^2 + \frac{1}{x^2} & y^2 + \frac{1}{y^2} \\ \frac{1}{y^2} + y^2 & x^2 + \frac{1}{x^2} \end{pmatrix} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

$$\det\begin{pmatrix} a-\lambda & b \\ b & a-\lambda \end{pmatrix} = 0 \qquad (a-\lambda)^2 - b^2 = 0$$

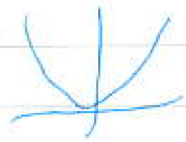
$$a - \lambda = \pm b$$

$$\lambda = a \pm b = (e^{2\beta} + e^{-2\beta}) \pm (e^{2\beta\varepsilon} + e^{-2\beta\varepsilon})$$

$$= 2(\cosh(2\beta) \pm \cosh(2\beta\epsilon))$$

$\cosh x \geq 0$

+ is bigger than −



so

$$\lambda(\varepsilon) = 2\left[\cosh(2\beta) + \cosh(2\beta\epsilon)\right]$$

$$\begin{bmatrix} a & \frac{1}{a} \\ \frac{1}{a} & a \end{bmatrix} \qquad \begin{bmatrix} b & \frac{1}{b} \\ \frac{1}{b} & b \end{bmatrix}$$

Date

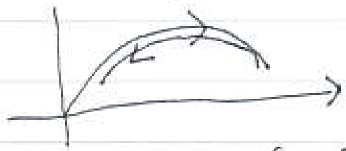
Microscopic laws are time-reversal ~~system~~ symmetric !

Newton's law $\quad m_i \frac{d^2\vec{r}_i}{dt^2} = \vec{F}(\vec{r}_1, \vec{r}_2 \dots \vec{r}_N)$

if $\vec{r}_i = \vec{r}_i(t)$ is a solution

so is $\vec{r}_i(-t)$

Newton's Eq. is invariant under $t \to -t$

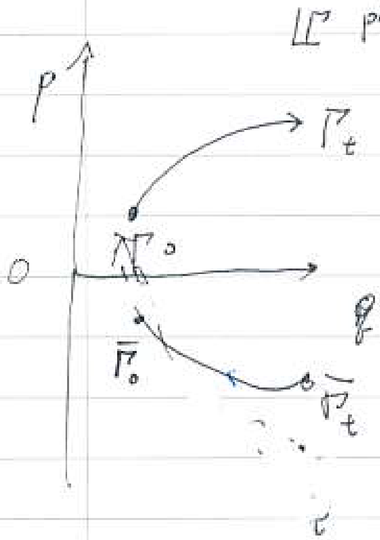


going forward & backward equally valid.

Hamiltonian Dynamics

$\Gamma_0 = (q_0, p_0)$ starting from Γ_0 we can move to $\Gamma_t = (q_t, p_t) = e^{t\mathcal{L}}\Gamma_0$ after time t.

$\mathcal{L} \cdot = [\cdot, H)$ poisson bracket



at Γ_t if we reverse the velocity $p_t \to -p_t$

when $\bar{\Gamma}_0 = e^{t\mathcal{L}}(q_t, -p_t) = (q_0, -p_0)$

continue running in time ~~forward~~

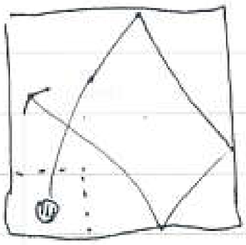
Quantum Mechanics

$\psi(t)$ is a solution of $i\hbar \frac{\partial \psi}{\partial t} = H\psi$

then $\psi^*(-t)$ is also a solution $\quad$ a solution moving back in time

Same is true for electromagnetic Equations of Maxwell.

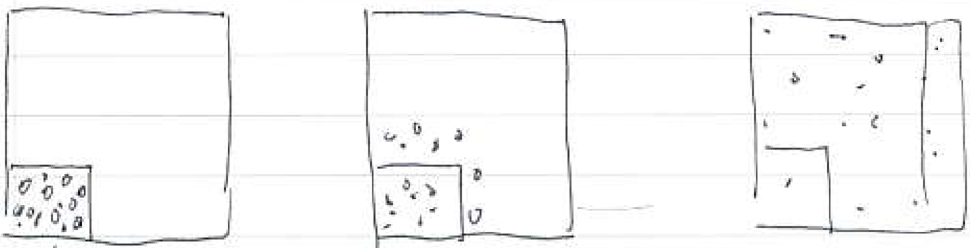
Date



one atom is a box at a box. $\rightarrow$ you cannot see a difference if the motion of the atom is reversed

(a movie played forward or backward, the is no difference)

$V > V_0$



$\nearrow V_0$ $t=0$ $t \to \infty$

many atoms of order 10^{23}

free expansion

$\mathcal{H} = \cancel{\mathcal{H}}\ U = \text{const}$ energy is conserved.

micro canonical

Boltzman's principle

$S = k \ln \Omega$

$\uparrow$ # of microscopic states

entropy (assume ideal gas)

$$S = N k_B \ln V + \underbrace{\text{terms depend on } U}_{\text{const if } U \text{ is fixed}}$$

[recall Sackur-Tetrode formula]

entropy increases by $\Delta S = N k_B \ln \frac{V}{V_0}$

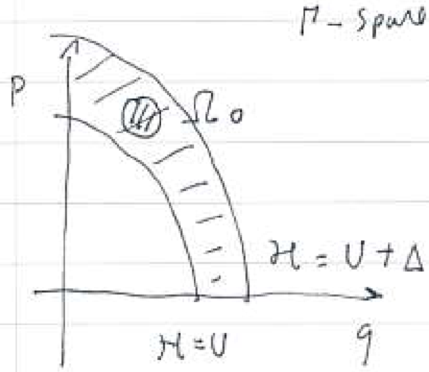
Reverse situation never happen no matter how long you wait. why? (because $N \to \infty$ limit) irreversibility due to the thermodynamic ?

it is easy to see if there is one or two atoms, it is frequent that the atoms go back to small box V_0 often!

$$T dS \geq dU + P dV = \delta Q$$

$\uparrow$ equal sign hold only in reversible process

Poincaré Recurrence Theorem (1892)



(0) the phase flow is one-to-one & onto

(1) phase space volume is finite

(2) starting in small region Ω_0 in phase space, because Liouville theorem the hypervolume does not change

(3) $\Omega_n = \hat{T}_t \Omega_{n-1}$ for some t

shown are later $\Omega_n \cap \Omega_0 \neq \phi$.

Ω_n is the set

(Consider $\Omega_0\ \Omega_1 \cdots \Omega_n$)

if they are all non intersecting, then as $n \to \infty$, the total phase volume goes to infinity

$$\sum_{i=0}^{n} |\Omega_i| = n|\Omega_0| \to \infty.$$ contradicts the assumption that the phase space is finite. (remember $S = k \ln \Omega$)

(in the sense of a set)

So the system returns to starting point arbitrarily close after t_p.

However $t_p \propto V^N \to \infty$ as $N \to \infty$.

t_p is too large (more than the age of Universe) ~~too~~ to be relevant.

However, if entropy is the functions of Γ, we have to conclude then S can ~~increase~~ decrease as well as increase due to Poincaré Recurrence!

Gibbs entropy

$$\int \rho(\Gamma)\, d\Gamma = 1$$

$$\bar{S} = -k_B \int \rho(\Gamma) \ln \rho(\Gamma)\, d\Gamma$$

Let assume we have a microscopic ensemble

$$\rho(\Gamma) = \begin{cases} C & U < H < U+\Delta \\ 0 & 0 \end{cases} \qquad C \int_{U<H<U+\Delta} d\Gamma = C\Omega = 1$$

then $\bar{S} = -k_B \int_{U<H<U+\Delta} C \ln C\, d\Gamma = -k_B \ln C = k_B \ln \Omega$

consistent with Boltzman entropy. in equilibrium.

$\bar{S}$ do not increase even in nonequilibrium situation

$\bar{S}$ = const for any $\rho(\Gamma, t)$

function of Γ and time ~~$\rho(\Gamma, t) = \rho(t)$~~

prove

$$\frac{d\bar{S}}{dt} = -k_B \int \frac{\partial}{\partial t}(\rho \ln \rho)\, d\Gamma$$

$$= -k_B \int \left[\frac{\partial \rho}{\partial t} \ln \rho + \rho \frac{\partial \ln \rho}{\partial t}\right] d\Gamma \qquad \frac{\partial \ln \rho}{\partial t} = \frac{1}{\rho}\frac{\partial \rho}{\partial t}$$

$$= -k_B \int \left(\frac{\partial \rho}{\partial t}\right)(\ln \rho + 1)\, d\Gamma$$

but $\frac{\partial \rho}{\partial t} + \{\rho, H\} = 0$ $\qquad \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial q}\frac{\partial \rho}{\partial p} = 0$

poisson bracket

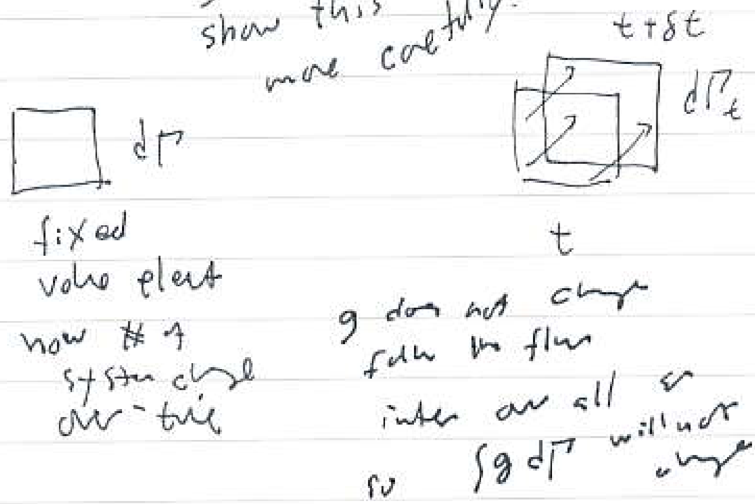
$$= -k_B \int \left[-\frac{\partial \rho}{\partial q}\dot{q} + \frac{\partial \rho}{\partial p}\dot{p}\right][\ln \rho + 1]\, d\Gamma$$

$$\frac{d\bar{S}}{dt} = -k_B \int \frac{\partial}{\partial t}(\rho \ln \rho)\, d\Gamma = -k_B \int \frac{D}{Dt}(\rho \ln \rho)\, d\Gamma = 0$$

because $\frac{D\rho}{\delta t} = 0$ by Liouville theorem

$$\int \frac{\partial}{\partial t} g\, d\Gamma = \int \frac{D}{Dt} g\, d\Gamma$$

show this more carefully.



how g change flowing the motion of particle.
however $d\Gamma_t = d\Gamma$ by Liouville theorem

$\bar{S}$ ~~sh~~ would not change for our gas experiment!

Same is true if we use Quantum $\hat{\rho}$ density matrix

$$\bar{S} = \mathrm{Tr}(\hat{\rho} \ln \hat{\rho})$$

$$\frac{d\bar{S}}{dt} = 0$$

This appear to say entropy is only meaningful for Equilibrium macroscopic system!!

proof of H-theorem

one particle distribution valid only for dilute gas

$$\rightarrow S_B = -k_B \int f \ln f \, d^3\vec{r}\, d^3\vec{p}$$

Boltzmann's version for reduced variable f

$= -k_B H$

$$f(\vec{r}_1) = N \int \rho(\vec{r}_1 \vec{r}_2 \ldots \vec{r}_N \vec{p}_1 \ldots \vec{p}_N)\, d\vec{r}_2 \ldots d\vec{p}_2 \ldots$$

$$H = \int f \ln f \, d\vec{r} d\vec{p}$$

Date

$$\frac{dH}{dt} = \int \frac{\partial}{\partial t}[f \ln f] \, d\vec{r} d\vec{p} = \int \left(\frac{\partial f}{\partial t} \ln f + \frac{\partial f}{\partial t} \right) d\vec{r} d\vec{p}$$

$$= \int \frac{\partial f}{\partial t} (1 + \ln f) \, d\vec{r} d\vec{p}$$

NO contrib to the integr

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{r}} + \vec{F} \cdot \frac{\partial f}{\partial p} = \left(\frac{\partial f}{\partial t} \right)_{col}$$

$\vec{v}$ and $\vec{r}$ are indep. variables $\quad \vec{v} = \frac{\vec{p}}{m}$

consider $\quad \int \frac{\partial}{\partial x}(f v_x) (1 + \ln f) \, d\vec{r} d\vec{p}$

$$= \int \frac{\partial}{\partial x} (v_x f \ln f) \, d\vec{r} d\vec{p} = \int dx \frac{\partial}{\partial x} v_x f \ln f \, dy dz d\vec{p}$$

$$- \int dy \, dz \, d\vec{p} \, [v f \ln f] \Big|_{x=\infty}^{x=-\infty} = 0$$

(due to $f = 0$ at the wall)

$$\frac{\partial}{\partial x} (f \ln f) = \frac{\partial f}{\partial x} \ln f + f \frac{1}{f} \frac{\partial f}{\partial x} = \frac{\partial f}{\partial x} (1 + \ln f)$$

assump $f \to 0$ when $|x| \to \infty$ or $|p_x| \to \infty$

same trick works for $\frac{\partial}{\partial p_x} F_x f$ as long as F is a function of $\vec{r}$ only (i.e. conservative force)

→ so

$$\frac{dH}{dt} = \int (1 + \ln f) \left(\frac{\partial f}{\partial t} \right)_{coll} d\vec{r} d\vec{p}$$

$\vec{p} = p$ to save writing

$$\left(\frac{\partial f}{\partial t} \right)_{coll} = \int d\vec{p}_1 \int d\vec{p}\,' d\vec{p}_1' \, \sigma(\vec{p}, \vec{p}_1 | \vec{p}\,' \vec{p}_1') (f' f_1' - f f_1)$$

$$\frac{dH}{dt} = \int d\vec{r} d\vec{p} d\vec{p}_1 \int d\vec{p}\,' d\vec{p}_1' \, \sigma(\vec{p} \vec{p}_1 | \vec{p}\,' \vec{p}_1') \, \cancel{1+f} \, \cancel{1+f} \, (1 + \ln f)(f' f_1' - f f_1) \, \cancel{(1+f}$$

$$\frac{dH}{dt} = \int dr \int dp \int dp_1 \int dp' \int dp_1' \, \underbrace{\sigma(pp_1|p'p_1')}_{\text{symmetric due permutation of prime}} (1+\ln f)(f'f_1' - ff_1)$$

$$\| \quad \sigma(p'p_1'|pp_1)$$

all p's are dummy integration variables

$$\int dp\,dp_1\,dp'\,dp_1' \; \sigma(pp_1|p'p_1')\, f(p')\, f(p_1')$$

$$\int dp'\,dp_1'\,dp\,dp_1 \; \sigma(p'p_1'|pp_1)\, f f_1$$

$$\sigma(pp_1|p'p_1')$$

& interchange $p \to p'$, $p_1 \to p_1'$

this becomes identical to the second "−" term.

need time-reversal plus space inversion

So

$$\frac{dH}{dt} = \int d\vec{r}\; dp\,dp_1\,dp'\,dp_1'\; \sigma(pp_1|p'p_1') \ln f\,(f'f_1' - ff_1)$$

① $\sigma(pp_1|p'p_1') \ln f\,(f'f_1' - ff_1) \;\; = \phi_1 = \phi$

interchange $p \leftrightarrow p_1$, $p' \leftrightarrow p_1'$ ② $\sigma(p_1p|p_1'p') \ln f_1\,(f_1'f' - f_1 f) \;\; = \phi_2 = \phi$

interchange prime & non-prime of above ③ $\sigma(p'p_1'|pp_1) \ln f'\,(ff_1 - f'f_1') \;\; = \phi_3 = \phi$

④ $\sigma(p_1'p'|p_1p) \ln f_1'\,(f_1f - f_1'f') \;\; = \phi_4 = \phi$

$\begin{cases} p \leftrightarrow p' \\ p \leftrightarrow p_1' \end{cases}$ $\sigma(\cdot\cdot)$ are all the same by symmetric $\qquad \ln 1 = 0$

add all $\phi = \frac{1}{4}(\phi_1 + \phi_2 + \phi_3 + \phi_4)$

$$\frac{dH}{dt} = \frac{1}{4}\int d\vec{r}\,dp\,dp_1\,dp'\,dp_1'\,\sigma(pp_1|p'p_1') \left[\ln\left(\overset{A}{ff_1}/\underset{B}{f'f_1'}\right)\cdot(\underset{B}{f'f_1'} - \underset{A}{ff_1}) \right]$$

$$= \int d\vec{r}\,dp\,dp_1\,dp'\,dp_1'\,\sigma(\cdot\cdot)\,(B-A)\ln\frac{A}{B}$$

$\sigma \geq 0$, $A \geq 0$, $B \geq 0$

if $A<B$, $B-A>0$, $\ln\frac{A}{B}<0$ $\quad (B-A)\ln\frac{A}{B}<0$

$A>B$ $\quad B-A<0$, $\ln\frac{A}{B}>0$ $\quad (B-A)\ln\frac{A}{B}<0$

so $\frac{dH}{dt} \leq 0 \qquad \dot{S}_B \geq 0$

Date

H - theorem $\frac{dH}{dt} \le 0$

i.e. $\int f \ln f \, d\vec{r} \, d\vec{p}$ is constant or decrese but never increase with time.

entropy then always non-decresing. entropy increase.

"Entropy cannot be a microscopic quantity"

Entropy concept appears (in the sense $\dot{S} > 0$) only at course-grain level. (macroscopic level)

Boltzmann Eq. describes irreversible process

For maxwell distribution $f = A e^{-\beta \frac{p^2}{2m}}$

1) from thermodynamic point of view entropy is unique & physical (because it is heat)

$\Delta S = \frac{\delta Q}{T}$ in reversal process

An: due to energy conservation

$$\frac{f f_1}{f' f_1'} = e^{-\beta\left(\frac{p^2}{2m} + \frac{p_1^2}{2m} - \frac{p'^2}{2m} - \frac{p_1'^2}{2m}\right)} = 1$$

2) Then is stat-mech entropy depends on the level of coarsing ?

so $\frac{dH}{dt} = 0$ i.e. coarse-grain Gibbs formula

i.e. Maxwell distribution is stationary.

equalibrium state $\frac{dH}{dt} = 0$ $f_0 f_1 - f' f_1' = 0$ → Maxwell distribution is a solution

if f is maxwell distribution then $\frac{dH}{dt} = 0$

3) what level of coarsing is good enough ?

Q4.

10 Aug 07

Renormalization group transformation in 1D

| a |

o x o x o x o x o . . before transfm

1 2 3 4 5 6 7 8 9

N spin

|—— 2a ——| $b=2$

decimation

o o o o o

1 3 5 7 . .

scale the system back to lattice constant a

o o o o

rename. 1 2 3 . $\frac{N}{2}$ spin

Real space RG

① find out the transform of H

② find fixed pt and linearized around fixed pt to find critical exponents

$$\bar{H} = -K\sum_{i=1}^{N} \sigma_i\sigma_{i+1} - h\sum_{i=1}^{N}\sigma_i - \sum_{i=1}^{N} C$$

← free energy

β absorbed in couplings

$$Z = \sum_{\{\sigma\}} e^{-\bar{H}}$$

Aim: to change the scale by a factor of 2.

trace out the even-numbered spins.

$$Z = \sum_{\{\sigma\}} e^{\left(K\sum_{i=1}^{N}\sigma_i\sigma_{i+1} + h\sum_{i=1}^{N}\sigma_i + \sum_{i=1}^{N} C\right)}$$

$$= \sum_{\{\sigma\}} \underbrace{e^{K\sigma_1\sigma_2 + h\sigma_1 + C}\, e^{K\sigma_2\sigma_3 + h\sigma_2 + C}}\, e^{K\sigma_3\sigma_4 + h\sigma_3 + C} \cdots$$

$$Z = \sum_{\{\sigma\}} \prod_{i=2,4,6} e^{\{K\sigma_i(\sigma_{i-1}+\sigma_{i+1}) + h\sigma_i + \frac{1}{2}h(\sigma_{i-1}+\sigma_{i+1}) + 2C\}}$$

Date

sum over σ_i for i even

$$= \sum_{\sigma_1 \sigma_3 \cdots \sigma_{i-1}} \prod_{i=2,4,6,} \left(e^{K(\sigma_{i-1}+\sigma_{i+1}) + h + \frac{h}{2}(\sigma_{i-1}+\sigma_{i+1}) + 2C} + e^{-K(\sigma_{i-1}+\sigma_{i+1}) - h + \frac{h}{2}(\sigma_{i-1}+\sigma_{i+1}) + 2C} \right)$$

relabeling the spin

$$Z' = \sum_{\{\sigma\}} \prod_{i=1}^{N/2} \left(e^{(K+\frac{h}{2})(\sigma_i+\sigma_{i+1}) + h + 2C} + e^{(-K+\frac{h}{2})(\sigma_i+\sigma_{i+1}) - h + 2C} \right)$$

We demand the renormalized partition function can be cast in the same form as the original

$$= \sum_{\{\sigma\}} \prod_i e^{(K'\sigma_i\sigma_{i+1} + h'\sigma_i + C')} \qquad i=1$$

symmetric fn

$$e^{K'\sigma_1\sigma_2 + \frac{h'}{2}(\sigma_1+\sigma_2) + C'}$$

$$= e^{(K+\frac{h}{2})(\sigma_1+\sigma_2) + h + 2C} + e^{(-K+\frac{h}{2})(\sigma_1+\sigma_2) - h + 2C}$$

$$\begin{aligned}
&\sigma_1 = \sigma_2 = +1 && e^{K'+h'+C'} = e^{(2K+2h+2C)} + e^{(-2K+2C)} && (1)\\
&\sigma_1 = \sigma_2 = -1 && e^{K'-h'+C'} = e^{(2K-2h+2C)} + e^{(-2K+2C)} && (2)\\
&\left.\begin{aligned}\sigma_1 = -\sigma_2 = 1\\ \sigma_1 = -\sigma_2 = -1\end{aligned}\right\} && e^{-K'+C'} = e^{h+2C} + e^{-h+2C} && (3)
\end{aligned}$$

$$(1) \div (2) \quad e^{2h'} = \frac{e^{2h}e^{2K} + e^{-2K}}{e^{-2h}e^{2K} + e^{-2K}} = e^{2h}\frac{\cosh(2K+h)}{\cosh(2K-h)}$$

$(1)\times(2)\div(3)^2 \rightarrow \quad e^{4K'} = \cosh(2K+h)\cosh(2K-h)/\cosh^2 h$

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$(1)\times(2)\times(3)^2 \rightarrow \quad e^{4C'} = e^{8C}\cosh(2K+h)\cosh(2K-h)\cosh^2 h$

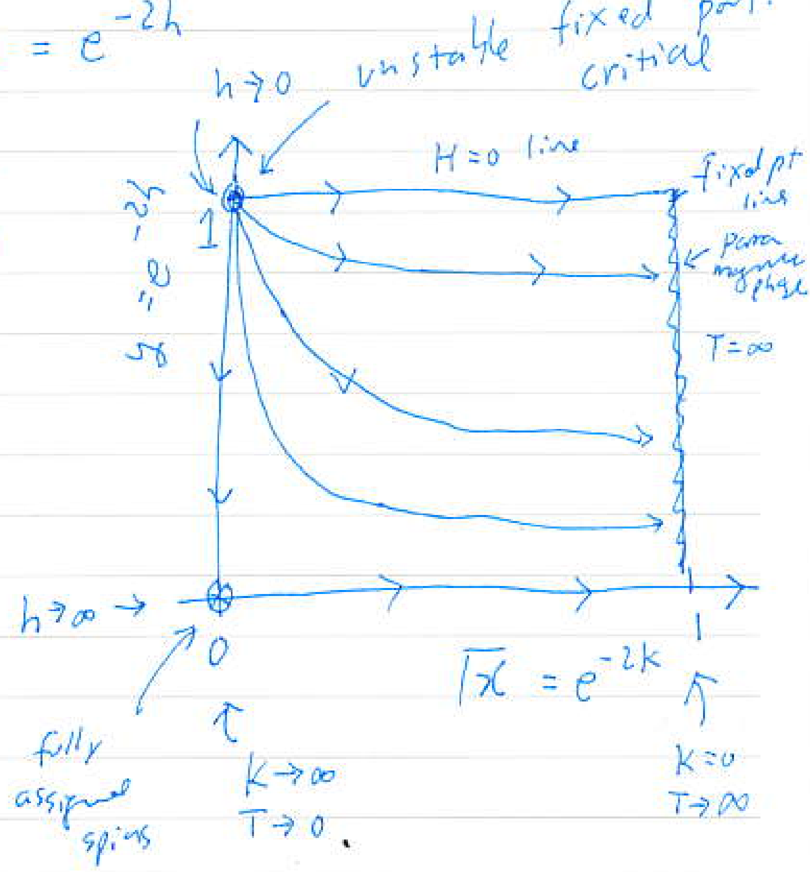
use new ~~at~~ variables $\quad w = e^{-4C}$

$x = e^{-4K}$

$y = e^{-2h}$

RG Transfm

$$\begin{cases} w' = w^2 \dfrac{x y^2 (1+y)^2}{(x+y)(1+xy)} \\[2ex] x' = \dfrac{x(1+y)^2}{(x+y)(1+xy)} \\[2ex] y' = \dfrac{y(x+y)}{(1+xy)} \end{cases}$$



linearized transformation at unstable fixed point $\quad (x^*, y^*) = (0,1)$ $\quad$ x small, ϵ small

consider $\quad x = x - x^* = x$ $\qquad y = 1+\epsilon$

$\epsilon = y - y^* = y-1$

$$x' = \frac{x(1+1+\epsilon)^2}{(x+1+\epsilon)(1+x(1+\epsilon))} \approx 4x = b^{?}x = 2^2 x = b^2 x$$

$$\epsilon' = y'-1 = \frac{(1+\epsilon)(x+1+\epsilon)}{1+x(1+\epsilon)} - 1 \approx$$

$$\frac{1+x+2\epsilon+\cdots}{1+x+\cdots} \approx 2\epsilon = b\epsilon$$

This means $\bar{f}_s(x, \epsilon) = b^{-1} \tilde{f}_s(b^2 x, b\epsilon)$ $\swarrow d=1$

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The usual variable has to be replaced by $x = e^{-4K}$!

Q5. $$\begin{cases} m\dfrac{dv}{dt} = -\dfrac{dV}{dx} - m\displaystyle\int_{-\infty}^{t} \gamma(t-t')\, v(t')\, dt' + R(t) \\ \dfrac{dx}{dt} = v \end{cases}$$ x, v

(a) $$\frac{d}{dt}E(t) = \frac{d}{dt}\left(\frac{1}{2}mv^2 + V\right) = \frac{1}{2}m\,2v\frac{dv}{dt} + \frac{dV}{dx}\frac{dx}{dt}$$

$$= \underline{m}v\underline{\dot{v}} + \frac{dV}{dx}v$$

$$= v\left[-\frac{dV}{dx} - m\int_{-\infty}^{t}\gamma(t-t')\,v(t')\,dt' + R(t)\right] + \frac{dV}{dx}v$$

$$= v(t)\left[-m\int_{-\infty}^{t}\gamma(t-t')\,v(t')\,dt' + R(t)\right]$$

(b) $$W_D = \int_0^T m\, v(t)\int_{-\infty}^{\infty}\gamma(t-t')\,v(t')\,dt'\; dt$$

$$W_R = \int_0^T v(t)\,R(t)\,dt$$

$$\int_0^T m\frac{dv}{dt}\cdot v\, dt = -\int_0^T \frac{dV}{dx}\cdot v(t)\,dt \; \cancel{+}\, W_D + W_R$$

$$\int_0^T \left[m v\dot{v} + \frac{dV}{dx}\cdot v\right]dt = -W_D + W_R$$

$$= \int_0^T \frac{dE(t)}{dt}\,dt = \int_0^T dE(t) = E(T) - E(0)$$ $\swarrow$ some bound value

$W_R \propto T$, $W_D \propto T$,

More precisely $\frac{1}{T}\left[E(T)-E(0)\right] = -\frac{W_D}{T} + \frac{W_R}{T} \to 0$ as $T\to\infty$

power = $\frac{\text{work}}{\text{time}}$ $\langle v\cdot R\rangle = \langle v\, m\int_{-\infty}^{t}\gamma(t-t')\,v(t)\,dt'\rangle$

$\langle\ \rangle = \frac{1}{T}\int_0^T dt$ avrge ~~rate of~~ power-dissipation.

problem set 6.3)

consider $\frac{d}{d\lambda}\left(e^{\lambda H}Ae^{-\lambda H}\right) = e^{\lambda H}HAe^{-\lambda H} + e^{\lambda H}(A(-H))e^{-\lambda H}$

$= e^{\lambda H}(HA-AH)e^{-\lambda H} = e^{\lambda H}[H,A]e^{-\lambda H} = \cancel{e^{\lambda H}}$

$= -i\hbar\, e^{\lambda H}\dot{A}e^{-\lambda H}$ $i\hbar\frac{dA}{dt} = [A,H] = i\hbar\dot{A}$ Heisenb eq

integrate both sides $\int_0^\beta d\lambda$

$\int_0^\beta d\lambda\frac{d}{d\lambda}\left(e^{\lambda H}Ae^{-\lambda H}\right) = -i\hbar\int_0^\beta d\lambda\, e^{\lambda H}\dot{A}e^{-\lambda H}$

$e^{\beta H}Ae^{-\beta H} - A = -i\hbar\int_0^\beta d\lambda\, e^{\lambda\beta}\dot{A}e^{-\lambda H}$ multiply ρ for left, B for right, take trace

$\beta\langle\dot{A};B\rangle = \cancel{\beta}\frac{1}{\cancel{\beta}}\int_0^\beta d\lambda\,\mathrm{Tr}\left(\rho e^{\lambda H}\dot{A}e^{-\lambda H}B\right)$

$\rho = \frac{e^{-\beta H}}{Z}$ $= \left(\frac{-1}{i\hbar}\right)\mathrm{Tr}\left[\rho e^{\beta H}Ae^{-\beta H}B - \rho AB\right]$

$= \frac{-1}{i\hbar}\mathrm{Tr}\left[\frac{1}{Z}Ae^{-\beta H}B - \rho AB\right]$

$= \frac{-1}{i\hbar}\mathrm{Tr}\left[\rho BA - \rho AB\right] = \frac{1}{i\hbar}\mathrm{Tr}\left[\rho(AB-BA)\right]$

$= \frac{1}{i\hbar}\langle[A,B]\rangle$

Assignment 6
problem 1

Date

$$
\begin{array}{c|c}
 & \begin{array}{cccc} \overset{\text{right}}{\rightrightarrows} & \rightleftarrows & \leftrightarrows & \leftleftarrows \end{array} \\
\text{left} & \\
\begin{array}{c} \rightrightarrows \\ \rightleftarrows \\ \leftrightarrows \\ \leftleftarrows \end{array} & P \begin{bmatrix} 1 & e^{-\beta\varepsilon} & e^{-\beta\varepsilon} & 0 \\ e^{-\beta\varepsilon} & 1 & 1 & e^{-\beta\varepsilon} \\ e^{-\beta\varepsilon} & 1 & 1 & e^{-\beta\varepsilon} \\ 0 & e^{-\beta\varepsilon} & e^{-\beta\varepsilon} & 1 \end{bmatrix}
\end{array}
$$

or

$$
P = \begin{bmatrix} 1 & x & x & 0 \\ x & 1 & 1 & x \\ x & 1 & 1 & x \\ 0 & x & x & 1 \end{bmatrix} \qquad x = e^{-\beta\varepsilon}
$$

$$|P - \lambda I| = 0$$

$$\lambda = 1,\ 0,\ \frac{1}{2}\left(3 \pm \sqrt{1+16x^2}\right)$$

$$\lambda_{max} = \frac{1}{2}\left(3 + \sqrt{1+16x^2}\right)$$

$$
\begin{aligned}
|P - \lambda I| &= \lambda^4 - 4\lambda^3 - 4x^2\lambda^2 + 5\lambda^2 + 4x^2\lambda - 2\lambda \\
&= (\lambda - 1)\lambda \cdot (2 - 4x^2 - 3\lambda + \lambda^2)
\end{aligned}
$$

$$
\begin{aligned}
F &= -k_BT \ln Z \\
&= -k_BT N \ln \lambda_{max}
\end{aligned}
$$

prob 2.

(a) $\xi(t,0) = b\,\xi(b^{y}t, 0)$ Let $b^{y}t = 1$

$b = t^{-\frac{1}{y}}$

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$= t^{-\frac{1}{y}} \cdots$

compare with $\xi \propto t^{-\nu} \rightarrow \nu = \frac{1}{y}$

(b) $M = \langle \sum_i \sigma_i \rangle = \frac{1}{Z} \sum_{\{\sigma\}} \sum \sigma_i \, e^{-\beta(E(\sigma) - h\sum_i \sigma_i)}$

$$\chi = \frac{\partial M}{\partial h}\Big|_{h\to 0} = \frac{1}{Z}\sum_{\{\sigma\}} \left(\sum \sigma_i\right)\left(\beta \sum_j \sigma_j\right) e^{-\beta H(\sigma)} - \frac{\langle \sum \sigma_i \rangle \langle \beta \sum \sigma_j \rangle}{Z^2}$$

$$= \beta \sum_{ij} \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle$$

$$= \beta \sum_{ij} G(|i-j|) = \beta \sum_i \sum_j G(j-i)$$

$$= \beta N \int d\vec{r}\, G(\vec{r})$$

(c)

$$\chi = \beta N \int d^d r\, G(r) \qquad \text{let } b = \xi$$

$$= \beta N \int r^{d-1} dr\, b^{(-d+2-\eta)} G(r/b, \xi/b)$$

$$= \beta N \xi^{-d+2-\eta} \int r^{d-1} dr\, G\left(\frac{r}{\xi}, 1\right)$$

$$= \beta N \xi^{-d+2-\eta} \xi^{d} \int_0^\infty \left(\frac{r}{\xi}\right)^{d-1} d\left(\frac{r}{\xi}\right) G\left(\frac{r}{\xi}, 1\right)$$

$$\approx \xi^{2-\eta} = t^{-\nu(2-\eta)}$$

compare with definition $\chi \approx t^{-\gamma}$

we find $\gamma = \nu(2-\eta)$!

prob.

3. $$m\frac{dv}{dt} = -m\gamma v + f + R(t)$$

Date

(a) $$m\frac{d\langle v\rangle}{dt} = -m\gamma\langle v\rangle + f$$

$$\frac{d\langle v\rangle}{dt} = -\gamma\langle v\rangle + f/m$$

$$\frac{d\langle v\rangle}{-\gamma\langle v\rangle + f/m} = dt$$

$$-\frac{1}{\gamma}\, d\left(\ln\left(-\gamma\langle v\rangle + \frac{f}{m}\right)\right) = dt$$

$$\ln\left(-\gamma\langle v\rangle + \frac{f}{m}\right) = -\gamma t + c$$

$$-\gamma\langle v\rangle + \frac{f}{m} = A e^{-\gamma t}$$

$$\langle v\rangle = \frac{f}{\gamma m} + A' e^{-\gamma t}$$

$$\langle v\rangle_0 = \frac{f}{\gamma m} + A'$$

$$\langle v\rangle = \frac{f}{\gamma m} + \left[\langle v_0\rangle - \frac{f}{\gamma m}\right] e^{-\gamma t}$$

if $R(t) \neq 0$

$$v(t) = \frac{f}{\gamma m} + \left[\langle v_0\rangle - \frac{f}{\gamma m}\right] e^{-\gamma t} + e^{-\gamma t}\int_0^t \frac{R(\tau)}{m} e^{\gamma\tau}\, d\tau$$

$$m\frac{dv(t)}{dt} = \left(-\gamma\langle v_0\rangle + f\right) e^{-\gamma t} - \gamma e^{-\gamma t}\int_0^t \frac{R(\tau)}{m} e^{\gamma\tau}\, d\tau + R(t)$$

$$= -m\gamma\left(v - \frac{f}{m\gamma}\right) + R(t)$$

$$v(t) - \bar{v}(t) = e^{-\gamma t} \int_0^t \frac{R(\tau)}{m} e^{\gamma\tau} d\tau$$

$$\langle (v(t) - \bar{v}(t))^2 \rangle = e^{-2\gamma t} \int_0^t d\tau\, d\tau' \frac{R(\tau) R(\tau')}{m^2} e^{\gamma(\tau+\tau')}$$

$$= e^{-2\gamma t} \int_0^t d\tau\, d\tau' \frac{1}{m^2} C\, \delta(\tau - \tau')\, e^{\gamma(\tau+\tau')}$$

$$= \frac{C e^{-2\gamma t}}{m^2} \int_0^t d\tau\, \cancel{e^{2\gamma\tau}}\, e^{2\tau\gamma}$$

$$= \frac{C}{m^2} e^{-2\gamma t} \left[\frac{1}{2\gamma} e^{2\tau\gamma} \right]_0^t$$

$$= \frac{C}{2\gamma m^2} e^{-2\gamma t} \left[e^{2t\gamma} - 1 \right]$$

$$= \frac{C}{2\gamma m^2} \left[1 - e^{-2\gamma t} \right]$$

$C = 2m\gamma k_B T$

$\left[\frac{k_B T}{m}\right]$

(3) Fockker-Panck equation

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial v}(M_1 P) + \frac{1}{2}\frac{\partial^2}{\partial v^2}(M_2 P)$$

$\tau \to 0$

$$M_1 = \frac{1}{\tau} \langle v(\tau) - v(0) \rangle$$

$$= \frac{1}{\tau}\left[\frac{f}{\gamma m} + \left(v(0) - \frac{f}{\gamma m}\right)(1-\gamma\tau) - v(0)\right]$$

$$M_2 = \frac{1}{\tau} \langle (v(\tau) - v(0))^2 \rangle$$

$$= \frac{1}{\tau}\left[\cancel{\frac{f}{\gamma m}} + \cancel{v} - \cancel{\frac{f}{\gamma m}} - \cancel{v} - \left(v - \frac{f}{\gamma m}\right)\gamma\tau\right]$$

$$= \frac{1}{\tau} \frac{C}{2\gamma m^2} \left[2\gamma\tau + \cdot \right]$$

$$= \frac{C}{m^2}$$

$$= -\left(v - \frac{f}{\gamma m}\right)\cdot\gamma$$

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v}\left[\gamma\left(v - \frac{f}{\gamma m}\right)P\right] + \frac{1}{2}\frac{C}{m^2}\frac{\partial^2}{\partial v^2}P$$

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$$\Xi = \sum_{n_1 n_2 \cdots n_N} e^{-\beta(\sum \varepsilon n_i - \mu n_i)}$$

$$= \sum_{n_1} e^{-\beta(\epsilon-\mu)n_1} \sum_{n_2} e^{-\beta(\epsilon-\mu)n_2} \cdots$$

$$= \left[\sum_{n=0}^{\infty} e^{-\beta(\epsilon-\mu)n}\right]^N = \left[\frac{1}{1-e^{-\beta(\epsilon-\mu)}}\right]^N$$

$$S = \frac{\partial}{\partial T}(k_B T \ln \Xi) = \frac{\partial}{\partial T} (-k_B T N \ln(1-e^{-\beta(\epsilon-\mu)})$$

$$\cancel{= -k_B N} = -N k_B \ln (1-e^{-\beta(\epsilon-\mu)})$$

$$- k_B T N \frac{e^{-\beta(\epsilon-\mu)}}{1-e^{-\beta(\epsilon-\mu)}} \left[\frac{\epsilon-\mu}{k_B T^2}\right]$$

$$= -N k_B \ln(1-e^{-\beta(\epsilon-\mu)}) - \frac{N(\epsilon-\mu)}{T} \frac{1}{e^{\beta(\epsilon-\mu)}-1}$$

$$\langle N \rangle = \frac{\partial}{\partial \mu}(k_B T \ln \Xi) = \cancel{k_B T \ln}$$

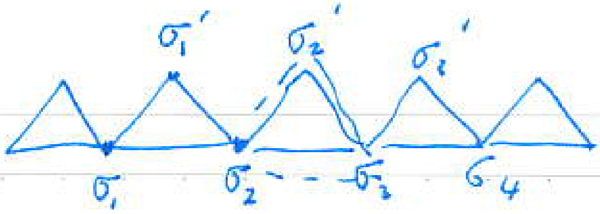
$$= \frac{\partial}{\partial \mu}\left[-N k_B T \ln(1-e^{-\beta(\epsilon-\mu)})\right]$$

$$= -N k_B \cancel{T} \frac{-e^{-\beta(\epsilon-\mu)}}{1-e^{-\beta(\epsilon-\mu)}} \cdot \cancel{\beta}$$

24 Apr 07

Date

3.



$$Z = \sum_{\{\sigma\}} \cdots e^{K(\sigma_2\sigma_3 + \sigma_2\sigma_2' + \sigma_3\sigma_2')} e^{K(\sigma_3\sigma_4 + \sigma_3\sigma_3' + \sigma_4\sigma_3')}$$

$$= \sum \underbrace{\sum_{\sigma_2'} e^{K(\sigma_2\sigma_3 + \sigma_2\sigma_2' + \sigma_3\sigma_2')}}_{P(\sigma_2\sigma_3)}$$

$$=$$

$$P(\sigma_2\sigma_3) = \sum_{\sigma_2'} e^{K(\sigma_2\sigma_3 + \sigma_2\sigma_2' + \sigma_3\sigma_2')}$$

$$= e^{K(\sigma_2\sigma_3 + \sigma_2 + \sigma_3)} + e^{K(\sigma_2\sigma_3 - \sigma_2 - \sigma_3)}$$

$$P = \begin{array}{cc} & \begin{array}{cc} \sigma_3 \; + & - \end{array} \\ \begin{array}{c} \sigma_2 \\ + \\ - \end{array} & \begin{bmatrix} e^{3K} + e^{-K} & 2e^{-K} \\ 2e^{-K} & e^{-K} + e^{3K} \end{bmatrix} \end{array}$$

$$\begin{vmatrix} e^{3K} + e^{-K} - \lambda & 2e^{-K} \\ 2e^{-K} & e^{3K} + e^{-K} - \lambda \end{vmatrix} = 0$$

$$(e^{3K} + e^{-K} - \lambda)^2 - 4e^{-2K} = 0$$

$$\lambda = e^{3K} + e^{-K} \pm 2e^{-K} = \begin{cases} e^{3K} + 3e^{-K} \\ e^{3K} - e^{-K} \end{cases}$$

$$F = -Nk_BT \ln Z = -Nk_BT \ln \lambda$$

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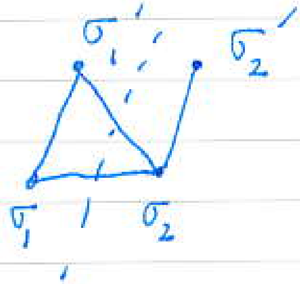
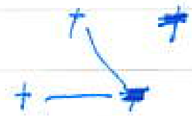
$$\lambda = e^{3K} + 3e^{-K}$$



$$e^{\frac{K}{2}\sigma_1\sigma_1'}$$



$$P(\sigma_1\sigma_1';\sigma_2\sigma_2') = e^{K\left(\frac{1}{2}\sigma_1\sigma_1' + \frac{1}{2}\sigma_2\sigma_2' + \sigma_1\sigma_2 + \sigma_1'\sigma_2'\right)}$$

$$= \begin{array}{c|cccc} \sigma_1\sigma_1' \backslash \sigma_2\sigma_2' & ++ & +- & -+ & -- \\ \hline ++ & e^{3K} & e^{2K} & e^{-2K} & e^{-K} \\ +- & 1 & e^{-K} & e^{-K} & 1 \\ -+ & 1 & e^{-K} & e^{-K} & 1 \\ -- & e^{-K} & e^{-2K} & e^{2K} & e^{3K} \end{array}$$

$$\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = e^{3K} - e^{-K},\ \lambda_4 = e^{3K} + 3e^{-K}$$

same as above

$$G = (T-T_c)\,aM^2 + bM^4$$

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$$h = \frac{\partial G}{\partial M} = (T-T_c)\,2aM + 4bM^3$$

$$\langle \tfrac{1}{2} m v^2 \rangle = \frac{k_B T}{2}$$

5. (a) $$P = \langle m\gamma v\cdot v \rangle = \gamma\langle m v^2\rangle = \gamma k_B T$$

(b) $$I = \langle R(t)\, v(t) \rangle$$

$$v(t) = \underset{\overset{\parallel}{0}}{v(0)\, e^{-\gamma t}} + \int_0^t \frac{R(\tau)}{m} e^{-\gamma(t-\tau)}\, d\tau$$

$t \to \infty$

$$I = \int_0^t \underbrace{\langle R(t) R(\tau) \rangle}_{C\delta(t-\tau)} \frac{e^{-\gamma(t-\tau)}}{m}\, d\tau$$

or $\delta(\tau - t)$

we have to think of δ is an idealization of gauss-like peak

integrate to t take half of the value

$$= \frac{C}{m}\int_0^t \delta(\tau - t)\, e^{-\gamma(t-\tau)}\, d\tau = \frac{C}{2m}$$

(c) Fluctuation-Dissipation give $C = 2m\gamma k_B T$

This shows $P = I$.

alternatively $$v m \frac{dv}{dt} = -m\gamma v^2 + R(t)\, v$$

$$\frac{d}{dt}\left(\tfrac{1}{2} m v^2\right) = \cdots$$

$$\frac{1}{T}\int_0^T (-m\gamma v^2)\, dt + \frac{1}{T}\int_0^T R(t)\, v\, dt$$

$$\frac{E(T) - E(0)}{T} = -\langle m\gamma v^2 \rangle + \langle R v \rangle = -P + I = 0$$

$T \to \infty$

entropy of ideal gas (monoatomic)

$$S = k_B N \ln(V U^{\frac{3}{2}}) + \text{const}$$

"depend on N but not on V & U"

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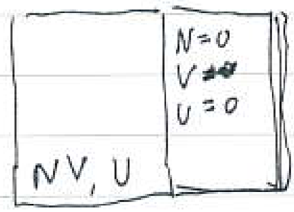
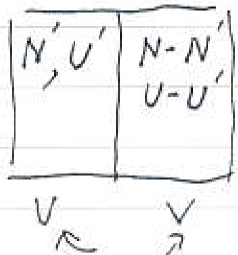
closed wall (adiabatic, immovable, ~~nonperm~~ impermeable)



→ compare



always fixed

S + 0

equal in volume

what is N' U' if partition is removed (becomes permeable)?

$$\cancel{S_{tot} = k_B N' \ln(V U'^{\frac{3}{2}}) + k_B (N-N') \ln(V(U-U')^{\frac{3}{2}})}$$

Need N-dependence

$$S = k_B N \ln(V U^{\frac{3}{2}}) + \frac{5}{2} N k_B (1 - \ln N) + \text{const}$$

$$S_{tot} = k_B N' \ln(V U'^{\frac{3}{2}}) + \frac{5}{2} N' k_B (1 - \ln N')$$

$$+ k_B (N-N') \underbrace{\ln(V(U-U')^{\frac{3}{2}})}_{k_B(N-N')[\ln V + \ln(U-U')^{\frac{3}{2}}]} + \frac{5}{2}(N-N') k_B (1 - \ln(N-N'))$$

$$= k_B N' \ln V + k_B N' \ln U'^{\frac{3}{2}} + k_B (N-N') \ln V + k_B(N-N') \ln(U-U')^{\frac{3}{2}}$$

$$+ \underbrace{\frac{5}{2} N' k_B + \frac{5}{2}(N-N') k_B} - \frac{5}{2} k_B N' \ln N' - \frac{5}{2} k_B (N-N') \ln(N-N')$$

$$= k_B N \ln V + \frac{5}{2} N k_B + k_B N' \ln \frac{U'^{3/2}}{\cancel{(U-U')}^{3/2}} + k_B(N-N') \ln(U-U')^{\frac{3}{2}}$$

$$- \frac{5}{2} k_B [N' \ln N' + (N-N') \ln(N-N')]$$

$$\frac{\partial S_{tot}}{\partial N'} = 0 = k_B \ln U'^{\frac{3}{2}} - k_B \ln(U-U')^{\frac{3}{2}}$$

$$- \frac{5}{2} k_B [\ln N' + \cancel{1} - \ln(N-N') - \cancel{1}]$$

$$\ln \underbrace{\frac{U'^{\frac{3}{2}}}{N'^{\frac{5}{2}}} \frac{(N-U')^{\frac{5}{2}}}{(U-U')^{\frac{3}{2}}}}_{\substack{\parallel \\ 1}} = 0 \qquad \left(\frac{N-N'}{N'}\right)^{\frac{5}{2}} \left(\frac{U'}{U-U'}\right)^{\frac{3}{2}} = 1$$

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$$\frac{\partial S_{tot}}{\partial U'} = \left[k_B N' \left[\frac{1}{U'^{3/2}}\right] + k_B (N-N') \frac{1}{(U-U')^{3/2}} (-1)\right] \frac{3}{2} = 0$$

$$\frac{N-N'}{N} = \frac{(U-U')^{\frac{3}{2}}}{U'^{3/2}} \rightarrow$$

$$\left(\frac{N-N'}{N'}\right)^{\frac{5}{2}} \cdot \left(\frac{N-N'}{N'}\right) = 1 \qquad \left(\frac{N-N'}{N'}\right)^{\frac{7}{2}} = 1$$

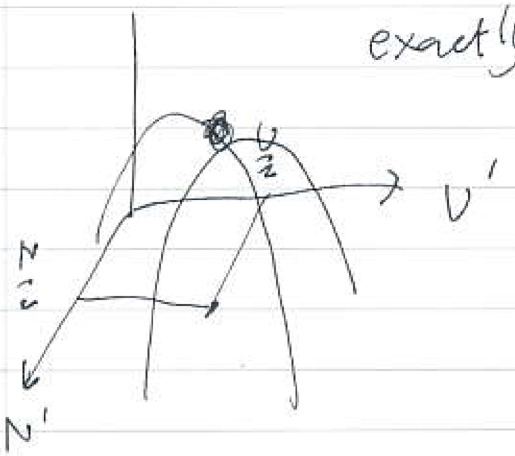
$$\frac{N-N'}{N'} = 1 \qquad N - N' = N' \qquad N = 2N'$$

$N' = \frac{N}{2}$ as expected.

$$\left(\frac{U-U'}{U'}\right)^{\frac{3}{2}} = 1 \qquad U - U' = U' \rightarrow U' = \frac{U}{2}$$

i.e. ~~partial~~ particles distributed evenly has max S.

exactly a maximum at $\left(\frac{N}{2}, \frac{U}{2}\right)$



12 Feb 08

$$\boxed{dU = TdS - pdV + \mu dN}$$

equilibrium statistical density operator $\hat{\rho}$

$\beta = \frac{1}{k_B T}$

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□ canonical $\hat{\rho} \propto e^{-\beta \hat{H}}$ $Tr(\hat{\rho}) = 1$ fix the constant

$Z = Tr(e^{-\beta \hat{H}}) \longrightarrow F = -k_B T \ln Z$

Helmholtz free energy

$dF = -SdT - pdV + \mu dN$

$\beta F = -\ln Z = \ln Z^{-1}$

$\hat{\rho} = \frac{1}{Z} e^{-\beta \hat{H}}$

$e^{\beta F} = Z^{-1}$

$= e^{-\beta(\hat{H} - F)}$

compare $F = U - TS$

$Tr \to \int \frac{dp_i dq_i \ldots}{h^{3N} N!}$

$\psi \to e^{-\beta(U-F)} = e^{-\beta(TS)}$

$= e^{-\frac{S}{k_B}}$ entropy

□ grand canonical $\hat{\rho} \propto e^{-\beta(\hat{H} - \mu \hat{N})}$

$\Xi = Tr\, e^{-\beta(\hat{H} - \mu \hat{N})}$

Fock space = 0 particle states ⊗ 1 particle Hilbert space ⊗ 2 particle Hilbert space ⊗ ...

$\hat{\rho} = \frac{1}{\Xi} e^{-\beta(\hat{H} - \mu \hat{N})}$ $Tr \hat{\rho} = 1$

$= e^{-\beta(\hat{H} - \mu \hat{N} - \Psi)}$ again

$e^{-\beta(U - \mu N - \Psi)}$

$\Psi = -k_B T \ln \Xi$

$= e^{-\beta(TS)} = e^{-\frac{S}{k_B}}$

$= F - \mu N = U - TS - \mu N = -pV$

$d\Psi = -SdT - pdV - Nd\mu$

□ isothermal - isobaric ensemble (fix T, P, N)

$G = F + pV = U - TS + pV = \mu N$ Gibbs free energy

$e^{-\beta(TS)} = e^{-\beta(U + pV - G)}$

$dG = -SdT + Vdp + \mu dN$

$\Downarrow$

$\hat{\rho} = e^{-\beta(\hat{H} + p\hat{V} - G)}$

$Tr \hat{\rho} = 1 \rightarrow G = -k_B T \ln Tr_{\hat{H}, \hat{V}} [e^{-\beta(\hat{H} + p\hat{V})}]$

classical case

$\rightarrow \int dV e^{-\beta pV}$

□ microcanonical $\hat{\rho} = \begin{cases} \sum |\varphi\rangle\langle\varphi| \cdot C & \leftarrow \text{sum over project} \\ \varphi \in (E, E+\Delta) \\ 0 \end{cases}$

$Tr|\varphi\rangle\langle\varphi| = 1$

$$Tr(\hat{\rho}) = 1 = C \; Tr \sum_{\varphi} |\varphi\rangle\langle\varphi| = C\,\Omega^{\text{Date}}$$ # of microstates

$C = \Omega^{-1}$

$=$

$$\hat{\rho} = \frac{1}{\Omega} \sum_{\varphi} |\varphi\rangle\langle\varphi|$$

$k_B \ln \Omega = S$

$\ln \Omega = S/k_B$

$$= e^{-S/k_B} \sum_{\varphi} |\varphi\rangle\langle\varphi|$$

$\Omega = e^{S/k_B}$

↑ also in this for if we take $\sum_{\varphi} |\varphi\rangle\langle\varphi| \approx 1$