PC5202 Advanced Statistical Mechanics

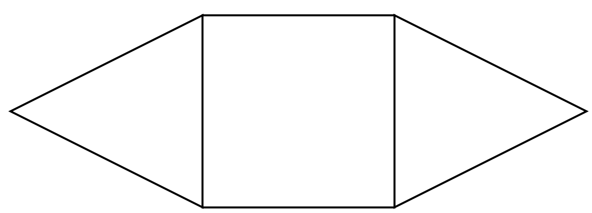
Assignment 5 (due Thursday, 2 April 2020)

1. Consider the Ising spins on a finite lattice (or planar graph) of one square and two triangles as shown in the figure below.

a) Draw the dual lattice of the given lattice. Give the number of links L, number of sites N, and number of faces F (plaquettes) in both the original lattice and the dual lattice and show that Euler's relation $N-L+F=2$ is satisfied.

b) Draw the diagrams which have a nonzero contribution to the partition function Z. Give the high-temperature series expansion of Z in variable $x = \tanh[J/(k_B T)]$.

c) Use the duality relation to find the low temperature expansion of the partition function Z^* on the dual lattice.



2. (3.15 of Plischke and Bergersen, modified). Consider an Ising chain with N spins $\sigma_i = \pm 1$ and periodic boundary conditions. The chain is coupled to an elastic field $-\infty < \varepsilon < +\infty$. Nonzero value of ε causes a dimerization of the chain, i.e. alternating bonds are strengthened (or weakened). The Hamiltonian for the system can, in reduced units, be written:

$$H = N\omega\varepsilon^2 - \sum_{i=1}^{N}\left[1-\varepsilon(-1)^i\right]\sigma_i\sigma_{i+1},$$

where $\sigma_{N+1}=\sigma_1$. The partition function is

$$Z_{tot} = \int_{-\infty}^{+\infty} d\varepsilon \sum_{\{\sigma_i\}} e^{-\beta H(\varepsilon,\sigma)}.$$

(a) Show that the partition function associated with the summation over the spins can be computed by the transfer matrix method

$$Z_\sigma = \mathrm{Tr}\left(PQ\right)^{\frac{N}{2}},$$

where

$$P = \begin{pmatrix} e^{\beta(1+\varepsilon)} & e^{-\beta(1+\varepsilon)} \\ e^{-\beta(1+\varepsilon)} & e^{\beta(1+\varepsilon)} \end{pmatrix}$$

is associated with even numbered sites and

$$Q = \begin{pmatrix} e^{\beta(1-\varepsilon)} & e^{-\beta(1-\varepsilon)} \\ e^{-\beta(1-\varepsilon)} & e^{\beta(1-\varepsilon)} \end{pmatrix}$$

corresponds to odd numbered sites.

(b) Let λ be the largest eigenvalue of the transfer matrix PQ. Show that the partition function for the whole system can then be written as $N \to \infty$

$$Z_{tot} = \int_{-\infty}^{+\infty} d\varepsilon\, e^{-\beta N g(\varepsilon)}$$

where $g(\varepsilon) = -\frac{k_B T}{2} \ln \lambda(\varepsilon) + \omega \varepsilon^2$.

(c) Show that the largest eigenvalue of the transfer matrix is

$$\lambda(\varepsilon) = 2\left[\cosh(2\beta) + \cosh(2\beta\varepsilon)\right].$$

3. Consider the standard two-dimensional nearest-neighbor Ising model on square lattice in the limit of zero magnetic field ($h > 0$, $h \to 0^+$):

$$E(\sigma) = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i .$$

(a) Compute the first three nonzero terms in a low-temperature series expansion of the total magnetization, $M = \lim_{h \to 0^+} \left\langle \sum_{i=1}^{N} \sigma_i \right\rangle$.

(b) Show that the result agrees with the exact answer of C. N. Yang,

$$\frac{M}{N} = \left(1 - \frac{1}{\sinh^4(2K)} \right)^{\frac{1}{8}},$$

when expanded in the low-temperature expansion variable e^{-2K}, $K=J/(k_B T)$, where N is the total number of spins.

[Extra bonus if you can compute more terms in (a) and (b).]

Tutorials:

4. Consider the Landau theory for ferromagnetic phase transitions. The Gibbs free energy as a function of temperature T and total magnetization M is assumed to be

$$G = (T - T_c)aM^2 + bM^4,$$

where a, b, and T_c are some positive constants.

(a) Give the corresponding expression for the Helmholtz free energy F as a function of temperature T and magnetic field h.

(b) Let $T = T_c$, show that $F(T_c, h) \propto h^{4/3}$.

(c) At $h = 0$, $T < T_c$, show that $F(T, 0) \propto (T - T_c)^2$.

(d) Assuming a scaling form for the Helmholtz free energy

$$F(t, h) = b^{-4} F(b^Y t, b^X h), \quad t = T - T_c,$$

and using the information obtained in (b) and (c), determine the scaling exponents X and Y.